

Tsirelson's problem and linear system games

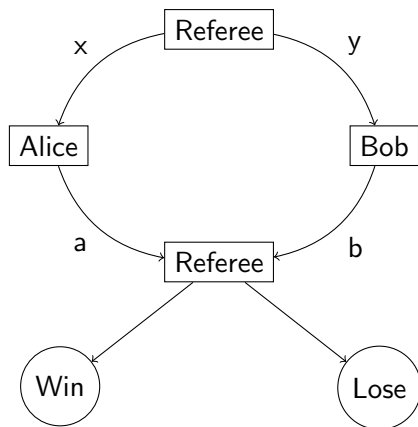
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includes joint work with Richard Cleve and Li Liu

Non-local games

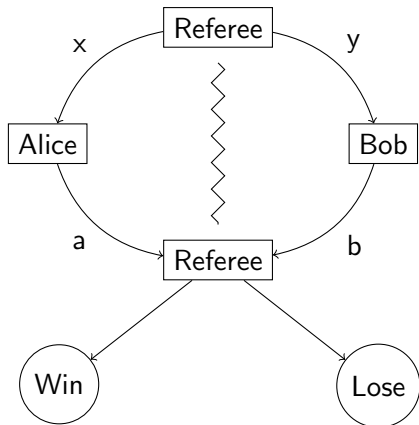


Win/lose based on outputs a, b and inputs x, y

Alice and Bob must cooperate to win

Winning conditions known in advance

Non-local games



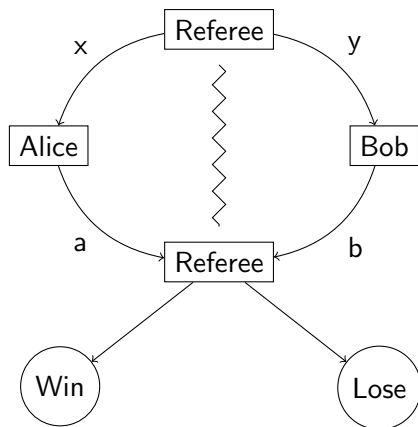
Win/lose based on outputs a, b and inputs x, y

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Winning conditions known in advance

Complication: players cannot communicate while the game is in progress

Strategies for non-local games



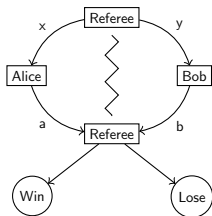
Suppose game is played many times, with inputs drawn from some public distribution π

To outside observer, Alice and Bob's strategy is described by:

$P(a, b|x, y)$ = the probability of output (a, b) on input (x, y)

Correlation matrix: collection of numbers $\{P(a, b|x, y)\}$

Classical and quantum strategies



$P(a, b|x, y)$ = the probability of output (a, b) on input (x, y)

Value of game ω = winning probability using strategy $\{P(a, b|x, y)\}$

What type of strategies might Alice and Bob use?

Classical: can use randomness, flip coin to determine output.

Correlation matrix will be $P(a, b|x, y) = A(a|x)B(b|y)$.

Quantum: Alice and Bob can share entangled quantum state

Bell's theorem: Alice and Bob can do better with an entangled quantum state than they can do classically

Quantum strategies

How do we describe a quantum strategy?

Use axioms of quantum mechanics:

- Physical system described by (finite-dimensional) Hilbert space
- No communication \Rightarrow Alice and Bob each have their own (finite dimensional) Hilbert spaces \mathcal{H}_A and \mathcal{H}_B
- Hilbert space for composite system is $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$
- Shared quantum state is a unit vector $|\psi\rangle \in \mathcal{H}$
- Alice's output on input x is modelled by measurement operators $\{M_a^x\}_a$ on \mathcal{H}_A
- Similarly Bob has measurement operators $\{N_b^y\}_b$ on \mathcal{H}_B

$$\text{Quantum correlation: } P(a, b|x, y) = \langle \psi | M_a^x \otimes N_b^y | \psi \rangle$$

Quantum correlations

Set of quantum correlations:

$$C_q = \left\{ \{P(a, b|x, y)\} : P(a, b|x, y) = \langle \psi | M_a^x \otimes N_b^y | \psi \rangle \text{ where} \right. \\ \left. \begin{array}{l} |\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B, \text{ where } \mathcal{H}_A, \mathcal{H}_B \text{ fin dim'l} \\ M_a^x \text{ and } N_b^y \text{ are projections on } \mathcal{H}_A \text{ and } \mathcal{H}_B \\ \sum_a M_a^x = I \text{ and } \sum_b N_b^y = I \text{ for all } x, y \end{array} \right\}$$

Two variants:

- 1 C_{qs} : Allow \mathcal{H}_A and \mathcal{H}_B to be infinite-dimensional
- 2 $C_{qa} = \overline{C_q}$: limits of finite-dimensional strategies

Relations: $C_q \subseteq C_{qs} \subseteq C_{qa}$

Commuting-operator model

Another model for composite systems: *commuting-operator model*

In this model:

- Alice and Bob each have an algebra of observables \mathcal{A} and \mathcal{B}
- \mathcal{A} and \mathcal{B} act on the joint Hilbert space \mathcal{H}
- \mathcal{A} and \mathcal{B} commute: if $a \in \mathcal{A}$, $b \in \mathcal{B}$, then $ab = ba$.

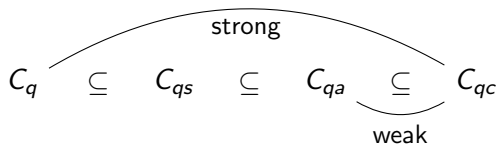
This model is used in quantum field theory

Correlation set:

$$C_{qc} := \left\{ \{P(a, b|x, y)\} : P(a, b|x, y) = \langle \psi | M_a^x N_b^y | \psi \rangle, \right. \\ \left. M_a^x N_b^y = N_b^y M_a^x \right\}$$

Hierarchy: $C_q \subseteq C_{qs} \subseteq C_{qa} \subseteq C_{qc}$

Tsirelson's problem



Two models of QM: tensor product and commuting-operator

Tsirelson problems: is C_t , $t \in \{q, qs, qa\}$ equal to C_{qc}

Fundamental questions:

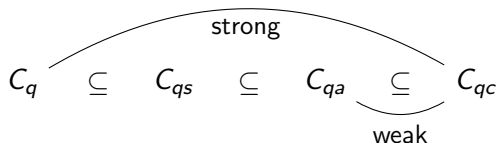
- 1 What is the power of these models?

Strong Tsirelson: is $C_q = C_{qc}$?

- 2 Are there observable differences between these two models, accounting for noise and experimental error?

Weak Tsirelson: is $C_{qa} = C_{qc}$?

What do we know?



Theorem (Ozawa, JNPPSW, Fr)

$C_{qa} = C_{qc}$ if and only if Connes' embedding problem is true

Theorem (S)

$C_{qs} \neq C_{qc}$

Other formulations (NPV)

Formulation due to Navascués, Pérez-García, Villanueva

Given $\{P(a, b|x, y)\}$

Local measurement statistics: $P(a|x) = \sum_{b,y} P(a, b|x, y)$,
 $P(b|y) = \text{similar}$

Rather than modeling joint system, model Bob's system:

- 1 For local measurement statistics, find measurements $\{N_b^y\}$ and density matrix ρ such that

$$P(b|y) = \text{tr}(N_b^y \rho)$$

- 2 For joint statistics, find measurements $\{N_b^y\}$ and density matrices ρ^{xa} such that

$$P(a, b|x, y) = P(a|x) \text{tr}(N_b^y \rho^{xa}).$$

Other formulations (NPV continued)

- 1 For local measurement statistics, find measurements $\{N_b^y\}$ and density matrix ρ such that

$$P(b|y) = \text{tr}(N_b^y \rho)$$

- 2 For joint statistics, find measurements $\{N_b^y\}$ and density matrices ρ^{xa} such that

$$P(a, b|x, y) = P(a|x) \text{tr}(N_b^y \rho^{xa}).$$

Question: Can Bob build a model of his local statistics which is consistent with Alice's observed inputs/outputs?

Answer: If and only if there are ρ^{xa} as above with $\sum_a P(a|x) \rho^{xa} = \rho$ (independent of x)

Other formulations (NPV continued)

Question: Can Bob build a model of his local statistics which is consistent with Alice's observed inputs/outputs?

Answer: If and only if there are ρ^{xa} as above with $\sum_a P(a|x)\rho^{xa} = \rho$ (independent of x)

Fact: This happens if and only if $\{P(a, b|x, y)\}$ belongs to C_{qs}

General state: a linear functional $f : \mathcal{B} \rightarrow \mathcal{H}$ such that $f(I) = 1$
and $f(A) \geq 0$ if A is positive

If ρ density matrix, then $f(A) = \text{tr}(A\rho)$ is general state

Not every general state comes from a density matrix

What if Bob uses general states instead of density matrices?

Other formulations (NPV continued)

Condition (*): Bob can build a model of his local statistics which is consistent with Alice's observed inputs/outputs

If Bob uses density matrices, then (*) holds if and only if $\{P(a, b|x, y)\}$ belongs to C_{qs}

If Bob uses general states, then (*) holds if and only if $\{P(a, b|x, y)\}$ belongs to C_{qc}

Conclusion:

Since $C_{qs} \neq C_{qc}$, modeling power of general states is greater than modeling power of density matrices, even for Bell scenarios

Other formulations (Ozawa)

Correlations with limited interactions:

$$C_{qc}(\epsilon) = \left\{ \{P(a, b|x, y)\} : P(a, b|x, y) = \langle \psi | M_a^x N_b^y | \psi \rangle \right. \\ \left. \begin{aligned} & \|M_a^x N_b^y - N_b^y M_a^x\| \leq \epsilon \\ & |\psi\rangle \in \text{finite-dim } \mathcal{H} \end{aligned} \right\}$$

These correlations are non-signalling

Theorem (Ozawa, Coudron-Vidick)

$$C_{qc} = \bigcap_{\epsilon > 0} C_{qc}(\epsilon)$$

If $\{P(a, b|x, y)\}$ has finite-dimensional limited interaction models for every $\epsilon > 0$, does it belong to C_q or C_{qa} ? (Answer: no)

Other fundamental questions

- ① Given a non-local game, can we compute the optimal value ω_t over strategies in C_t , $t \in \{qa, qc\}$?
- ② Is $C_q = C_{qa}$? (In other words, does every non-local game have an optimal finite-dimensional strategy?)
- ③ Given $P \in C_q$, is there a computable upper bound on the dimension needed to realize P ?

What do we know?

Theorem (Navascués, Pironio, Acín)

Given a non-local game, there is a hierarchy of SDPs which converge in value to ω_{qc}

Problem: no way to tell how close we are to the correct answer

Theorem (S)

It is undecidable to tell if $\omega_{qc} < 1$

General cases of other questions completely open!

Two theorems

Theorem (S)

$$C_{qs} \neq C_{qc}$$

Theorem (S)

It is undecidable to tell if $\omega_{qc} < 1$

Proofs: make connection to group theory via linear system games

Linear system games

Start with $m \times n$ linear system $Ax = b$ over \mathbb{Z}_2

Inputs:

- Alice receives $1 \leq i \leq m$ (an equation)
- Bob receives $1 \leq j \leq n$ (a variable)

Outputs:

- Alice outputs an assignment a_k for all variables x_k with $A_{ik} \neq 0$
- Bob outputs an assignment b_j for x_j

They win if:

- $A_{ij} = 0$ (assignment irrelevant) or
- $A_{ij} \neq 0$ and $a_j = b_j$ (assignment consistent)

Quantum solutions of $Ax = b$

Observables X_j such that

- ① $X_j^2 = I$ for all j
- ② $\prod_{j=1}^n X_j^{A_{ij}} = (-I)^{b_i}$ for all i
- ③ If $A_{ij}, A_{ik} \neq 0$, then $X_j X_k = X_k X_j$

(We've written linear equations multiplicatively)

Theorem (Cleve-Mittal, Cleve-Liu-S)

Let G be the game for linear system $Ax = b$. Then:

- G has a perfect strategy in C_{qs} if and only if $Ax = b$ has a finite-dimensional quantum solution
- G has a perfect strategy in C_{qc} if and only if $Ax = b$ has a quantum solution

Quantum solutions ct'd

The *solution group* Γ of $Ax = b$ is the group generated by X_1, \dots, X_n, J such that

- 1 $X_j^2 = [X_j, J] = J^2 = e$ for all j
- 2 $\prod_{j=1}^n X_j^{A_{ij}} = J^{b_i}$ for all i
- 3 If $A_{ij}, A_{ik} \neq 0$, then $[X_j, X_k] = e$

where $[a, b] = aba^{-1}b^{-1}$, $e =$ group identity

Theorem (Cleve-Mittal, Cleve-Liu-S)

Let G be the game for linear system $Ax = b$. Then:

- G has a perfect strategy in C_{qs} if and only if Γ has a finite-dimensional representation with $J \neq I$
- G has a perfect strategy in C_{qc} if and only if $J \neq e$ in Γ

Groups and local compatibility

Suppose we can write down any group relations we want...

But: generators in the relation will be forced to commute!

Call this condition *local compatibility*

Local compatibility is (a priori) a very strong constraint

For instance, S_3 is generated by a, b subject to the relations

$$a^2 = b^2 = e, (ab)^3 = e$$

If $ab = ba$, then $(ab)^3 = a^3b^3 = ab$

So relations imply $a = b$, and S_3 becomes \mathbb{Z}_2

Group embedding theorem

Solution groups satisfy local compatibility

Nonetheless:

Theorem (S)

Let G be any finitely-presented group, and suppose we are given J_0 in the center of G such that $J_0^2 = e$.

Then there is an injective homomorphism $\phi : G \hookrightarrow \Gamma$, where Γ is the solution group of a linear system $Ax = b$, with $\phi(J_0) = J$.

Furthermore, if X_1, \dots, X_n are some elements of G with $X_i^2 = e$, then we can also require that $\phi(X_i)$ is a generator of Γ .

Non-residually finite groups

Embedding theorem useful because there are groups with interesting properties

For instance, there are finitely-presented non-residually-finite groups:

K with an element $g \neq e$ such that $g \mapsto I$ in every finite-dimensional representation

For example, Higman's group:

$$K = \langle a, b, c, d : aba^{-1} = b^2, bcb^{-1} = c^2, \\ cdc^{-1} = d^2, dad^{-1} = a^2 \rangle$$

Only finite-dimensional representation is the trivial representation!

Strong Tsirelson is false

Start with group K with an element $g \neq e$ such that $g \mapsto I$ in every finite-dimensional representation

Add two generators x and J_0

Add relations $[g, x] = J_0$ and $[J_0, G] = J_0^2 = 1$.

Conclusion: get a group G with a central element $J_0 \neq e$, $J_0^2 = e$, such that $J_0 \mapsto I$ in every finite-dimensional representation

Embedding theorem: embed G in a solution group Γ

$$G \hookrightarrow \Gamma \rightarrow \mathcal{U}(n)$$

$$J_0 \mapsto J \mapsto I$$

Get a solution group Γ where $J \neq e$, but $J \mapsto I$ in every finite-dimensional representation

Strong Tsirelson is false (continued)

Get a solution group Γ where $J \neq e$, but $J \mapsto I$ in every finite-dimensional representation

Theorem (Cleve-Mittal, Cleve-Liu-S)

Let G be the game for linear system $Ax = b$. Then:

- G has a perfect strategy in C_{qs} if and only if Γ has a finite-dimensional representation with $J \neq I$*
- G has a perfect strategy in C_{qc} if and only if $J \neq e$ in Γ*

Game associated to Γ has a perfect strategy in C_{qc}

Does not have a perfect strategy in C_{qs}

Conclusion: $C_{qs} \neq C_{qc}$

How do we prove the embedding theorem?

Linear system $Ax = b$ over \mathbb{Z}_2 equivalent to labelled hypergraph:

Edges are variables

Vertices are equations

v is adjacent to e if and only if $A_{ve} \neq 0$

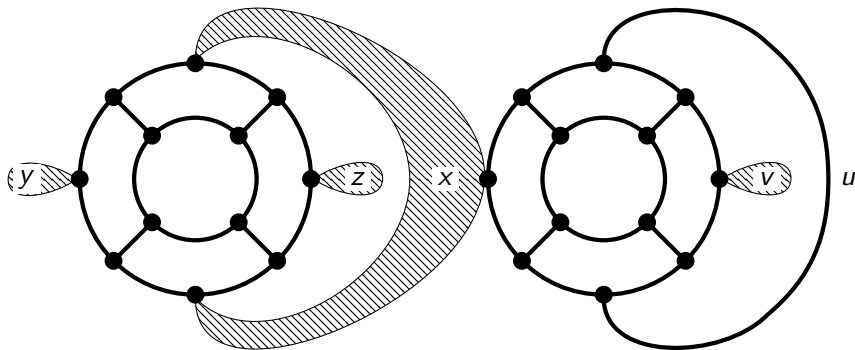
v is labelled by $b_i \in \mathbb{Z}_2$

Given finitely-presented group G , we get Γ from a linear system

But what linear system?

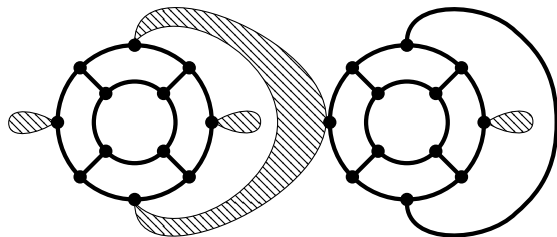
Can answer this pictorially by writing down a hypergraph?

The hypergraph by example



$$\langle x, y, z, u, v : xyxz = xuvu = e = x^2 = y^2 = \dots = v^2 \rangle$$

The end



$$\langle x, y, z, u, v : xyxz = xuvu = e = x^2 = y^2 = \dots = v^2 \rangle$$

Thank-you!