Perfect Embezzlement

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Based on:
Perfect Embezzlement of Entanglement(R. Cleve, L. Liu, V. Paulsen)
A non-commutative unitary analogue of Kirchberg's conjecture(S. Harris)

Outline

- Van Dam and Hayden Approximate Embezzlement
- Impossibility of Perfect Embezzlement in Tensor Framework
- Commuting Framework
- The C*-algebra of Non-commuting Unitaries
- Perfect Embezzlement
- New Versions of Tsirelson, Connes, and Kirchberg
- The Coherent Embezzlement Game

Approximate Embezzlement of A Bell State

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$$|0\rangle_A|0\rangle_B \otimes \psi \longrightarrow \frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B) \otimes \psi_\epsilon$$

where $\|\psi - \psi_\epsilon\| < \epsilon$ for any $\epsilon > 0$.

More precisely, given $\mathcal{H}_A = \mathcal{H}_B = \mathbb{C}^2$, there are finite dimensional spaces $\mathcal{R}_A, \mathcal{R}_B$ and unitaries, U_A on $\mathcal{H}_A \otimes \mathcal{R}_A$, U_B on $\mathcal{R}_B \otimes \mathcal{H}_B$ such that on $(\mathcal{H}_A \otimes \mathcal{R}_A) \otimes (\mathcal{R}_B \otimes \mathcal{H}_B)$,

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We now show why perfect embezzlement is impossible, in this tensor product framework.

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Proof: Write a Schmidt decomposition

$$|0\rangle \otimes \psi \otimes |0\rangle = \sum_j t_j(|0\rangle \otimes u_j) \otimes (v_j \otimes |0\rangle),$$

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But the Schmidt coefficients of $\frac{1}{\sqrt{2}}(|0\rangle \otimes \psi \otimes |0\rangle + |1\rangle \otimes \psi \otimes |1\rangle)$ are $\frac{t_1}{\sqrt{2}}, \frac{t_1}{\sqrt{2}}, \frac{t_2}{\sqrt{2}}, \frac{t_2}{\sqrt{2}}, \ldots$.

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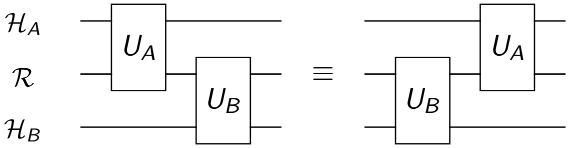

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In the rest of this talk, I want to outline the proof and show why the fact that perfect embezzlement is possible in this commuting framework but not possible in a tensor product framework is closely related to the Tsirelson conjectures and to Connes' embedding conjecture.

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Lemma

$(U_A \otimes id_B)$ commutes with $(id_A \otimes U_B)$ if and only if $U_{i,j}V_{k,l} = V_{k,l}U_{i,j}$ and $U_{i,j}^* V_{k,l} = V_{k,l}U_{i,j}^*$ for all i, j, k, l.

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Thus, we see that having commuting operator frameworks as above is exactly the same as having operator matrices $U_A = (U_{i,j})$ and $U_B = (V_{k,l})$ that yield unitaries and whose entries pairwise *-commute.

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Thus, a representation of $U_{nc}(n) \otimes_{max} U_{nc}(m)$ corresponds to operators $U_{i,j}, V_{k,l}$ where the $U_{i,j}$'s *-commute with the $V_{k,l}$'s such that $(U_{i,j})$ and $(V_{k,l})$ are unitary operator matrices.

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Recall that a *state* on a C*-algebra is just a positive linear functional s with $s(1) = 1$.

Theorem (CLP)

Perfect embezzlement of a Bell state is possible in a commuting operator framework if and only if there is a state s on $U_{nc}(2) \otimes_{max} U_{nc}(2)$ *satisfying* $s(u_{00} \otimes v_{00}) = s(u_{10} \otimes v_{10}) = 1/\sqrt{2}$ *and* $s(u_{00} \otimes v_{10}) = s(u_{10} \otimes v_{00}) = 0$.

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Corollary

The van Dam–Hayden approximate embezzlement results imply that there exists a state on $U_{nc}(2) \otimes_{min} U_{nc}(2)$ *satisfying the above equations.*

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Corollary

The van Dam–Hayden approximate embezzlement results imply that there exists a state on $U_{nc}(2) \otimes_{min} U_{nc}(2)$ *satisfying the above equations. Hence, the conditions of the above result are met and perfect embezzlement of a state is possible in a commuting operator framework.*

The representation of $U_{nc}(2) \otimes_{min} U_{nc}(2)$ given by the Corollary can not decompose as a spatial tensor product of a representation of each factor or else we would contradict the fact that perfect embezzlement is impossible in a tensor product framework!

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We now want to draw an analogy with quantum correlation matrices.

Tsirelson, Connes and all that

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$$p(x, y|a, b) = \langle \psi | E_{a,x} \otimes F_{y,b} | \psi \rangle.$$

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Here are some of the things that we know/don't know about these sets.

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- (JNPPSW + Ozawa)$C_q(n,m)^- = C_{qc}(n,m), \forall n,m$ iff Connes' Embedding conjecture has an affirmative answer.
- (Slofstra, April 2016) there exists an n,m(very large) such that $C_{qs}(n,m) \neq C_{qc}(n,m)$.

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The following are equivalent.

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The equivalence of the first three, is the analogue of Kirchberg's theorem relating Connes to tensor products of free group C*-algebras. The equivalence of the first and last is the analogue of the result of [Junge $\cdots$ Ozawa].

Unitary Correlation Sets

We set

$$UC_q(n,m) = \{\langle\psi|X \otimes Y|\psi\rangle : (U_{i,j}), (V_{k,l}) \text{ are unitary,}$$
$$U_{i,j} \in M_p, V_{k,l} \in M_q, \exists p, q, ||\psi|| = 1\} \subset M_n \otimes M_m.$$
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Here are some of the things that we know/don't know about these sets.

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- $UC_{qs}(2,2) \neq UC_{qc}(2,2)$.
- (Harris) $UC_q(n,m)^- = UC_{qc}(n,m), \forall n, m \iff$ Connes Embedding is true.

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The Coherent Embezzlement Game

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Alice and Bob jointly receive one of two states, ϕ_0, ϕ_1 where

$$\phi_c = \frac{1}{\sqrt{2}}|00\rangle \otimes |00\rangle + \frac{1}{\sqrt{2}}(-1)^c\big(\frac{1}{\sqrt{2}}|10\rangle \otimes |01\rangle + \frac{1}{\sqrt{2}}|11\rangle \otimes |11\rangle\big),$$

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Alice receives the first two qubits which is $\mathcal{H}_A$ and Bob receives the second two qubits, $\mathcal{H}_B$. They each output a classical bit a, b.
They win if input $\phi_0 \implies a + b = 0$, and input $\phi_1 \implies a + b = 1$.

Assume that they are allowed to share a state $\psi \in \mathcal{R}$ and act with unitaries on $\mathcal{H}_A \otimes \mathcal{R}$ and $\mathcal{R} \otimes \mathcal{H}_B$, respectively, where necessarily these unitaries commute.

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There is a perfect strategy for the coherent embezzlement game in the commuting framework. But there is no perfect strategy if we require that $\mathcal{R} = \mathcal{R}_A \otimes \mathcal{R}_B$ and that their unitaries act locally, even when we allow $\mathcal{R}_A$ and $\mathcal{R}_B$ to be infinite dimensional.

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Idea of proof: 1)This game is embezzlement in reverse!
2) Unitaries are reversible, i.e., invertible.

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Maybe embezzlement will give us a way to swindle a solution to these problems!

Thanks!