Semidefinite approximations for optimization problems in quantum information

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Plan of the first two lectures

- **Today:**

  "Classical" polynomial optimization: build SDP relaxations using sum-of-squares polynomials and moment matrices.

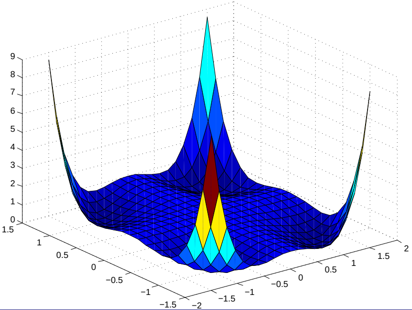
- **Tomorrow:**

  Extension to polynomial optimization in non-commutative variables.

- Application to quantum information: SDP hierarchies for the entangled value of non-local games and the maximum violation of Bell inequalities.

Classical polynomial optimization

What is Polynomial Optimization?



(P) *Minimize a* **polynomial** *function* p *over a region*

$$K = \{x \in \mathbb{R}^n \mid h_1(x) \geq 0, \ldots, h_m(x) \geq 0\}.$$

defined by **polynomial inequalities.** *Compute:*

$$p_{\min} := \min_{x \in K} p(x).$$

$p, h_1, \ldots, h_m \in \mathbb{R}[\mathbf{x}] = \mathbb{R}[x_1, \ldots, x_n]$ are multivariate polynomials.

Some hard instances

(P) is hard for simple sets K: $\mathbb{R}^n$, $\{\pm 1\}^n$, simplex, cube, sphere

- Deciding if the sequence $a_1, \ldots, a_n$ can be partitioned, i.e., if $\sum_{i \in S} a_i = \sum_{i \in [n] \setminus S} a_i$ for some $S \subseteq [n]$, is NP-complete.

  It is equivalent to checking whether $p_{\min} = 0$, where

$$p_{\min} = \min \left(\sum_{i=1}^{n} x_i a_i\right)^2 \;\text{ s.t. }\; x_1, \ldots, x_n \in \{\pm 1\},$$

$$\text{or} \quad p_{\min} = \min \left(\sum_{i=1}^{n} x_i a_i\right)^2 + \sum_{i=1}^{n} (1 - x_i^2)^2 \;\text{ s.t. }\; x \in \mathbb{R}^n.$$

- Computing the **stability number** $\alpha(G)$ of graph $G = (V, E)$:

$$\alpha(G) = \max \sum_{v \in V} x_v \;\text{ s.t. }\; x_u x_v = 0 \; (uv \in E),\; x_u \in \{0, 1\} \; (u \in V)$$

$$\frac{1}{\alpha(G)} = \min \; x^T (I + A_G) x \;\text{ s.t. }\; \sum_{v \in V} x_v = 1,\; x_v \geq 0 \; (v \in V)$$

Strategy for polynomial optimization

Shor [1987], Nesterov, Lasserre, Parrilo [2000 –]

Approximate (P) by a hierarchy of convex (semidefinite) relaxations.

Such relaxations can be constructed using

representations of positive polynomials as sums of squares of polynomials

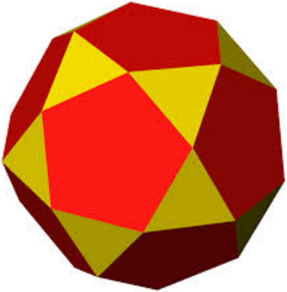
and

the dual theory of moments

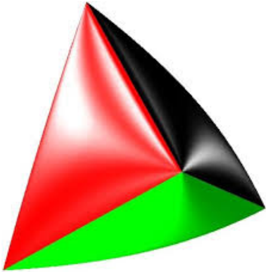
Linear Programming & Semidefinite Programming

Optimize a linear function over

a polyhedron	a convex set (spectrahedron)
$\langle a_j, x\rangle = b_j,\ x \geq 0$	$\langle A_j, X\rangle = b_j,\ X \succeq 0$
vector variable	matrix variable



LP



SDP

SDP is linear optimization over the cone of positive semidefinite matrices. There are **efficient algorithms** to solve LP & SDP.

Semidefinite programming

Trace inner product: $\langle C, X \rangle = Tr(CX) = \sum_{i,j=1}^{n} C_{ij} X_{ij}$.

(Primal semidefinite program)

Given symmetric matrices $C, A_j \in \mathbb{R}^{n \times n}$ *and* $b \in \mathbb{R}^m$, *compute:*

$$p^* := \sup \ \langle C, X \rangle \ \ \textit{such that} \ \langle A_j, X \rangle = b_j \ (j = 1, \ldots, m), \ \ X \succeq 0.$$

(Dual semidefinite program)

$$d^* := \inf \ \sum_{j=1}^{m} b_j y_j \ \ \textit{such that} \ \sum_{j=1}^{m} y_j A_j - C \succeq 0.$$

Theorem

- *Weak duality:* $p^* \leq d^*$.
- *Strong duality: If the dual is* **bounded** $(d^* > -\infty)$ *and* **strictly feasible** $(\sum_j y_j A_j - C \succ 0$ *for some* $y)$, *then* $p^* = d^*$ *and there is a primal optimal solution.*

Strategy for polynomial optimization

$$K = \{x \in \mathbb{R}^n \mid h_1(x) \geq 0, \ldots, h_m(x) \geq 0\}.$$

$$p_{\min} = \min_{x \in K} p(x) = \sup \ \lambda \ \text{ s.t. } \ p(x) - \lambda \geq 0 \ \forall x \in K$$

$$p_{\text{sos}} := \sup \ \lambda \text{ s.t. } p - \lambda = s_0 + s_1 h_1 + \ldots + s_m h_m.$$

where s_j are sum-of-squares polynomials.

$$p_{\text{sos}} \leq p_{\min}.$$

Testing whether a polynomial p is nonnegative is **hard**, *but one can test whether p is a sum of squares of polynomials:* $p = \sum_i s_i^2$ *for some polynomials* s_i,

efficiently using semidefinite programming.

Recognizing sums of squares of polynomials with SDP

Gram-matrix method of Powers-Wörmann [1998]:

$$p(x) = \sum_{|\alpha| \le 2d} p_\alpha x^\alpha$$ is a **sum of squares of polynomials**

$$\Updownarrow$$

$$p(x) = \sum_i s_i(x)^2 \qquad [\text{ write } s_i(x) = \overline{s_i}^T [x]_d \ ]$$

$$\Updownarrow$$

$$p(x) = \sum_i [x]_d^T \overline{s_i}\ \overline{s_i}^T [x]_d = [x]_d^T \underbrace{\left(\sum_i \overline{s_i}\ \overline{s_i}^T \right)}_{X \succeq 0} [x]_d$$

$$\Updownarrow$$

The SDP: $\left\{ \begin{array}{l} \sum_{\beta,\gamma|\beta+\gamma=\alpha} X_{\beta,\gamma} = p_\alpha \quad (|\alpha| \le 2d) \\ X \succeq 0 \end{array} \right.$ **is feasible**

Example: $p(x,y) = x^4 + 2x^3y + 3x^2y^2 + 2xy^3 + 2y^4$ SOS?

$$p(x,y) = (x^2\ xy\ y^2)\ \underbrace{\begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}}_{X \succeq 0 \rightsquigarrow X = \sum_i u_i u_i^T} \begin{pmatrix} x^2 \\ xy \\ y^2 \end{pmatrix}$$

Equate coefficients on both sides:

x^4: $a = 1$ x^3y: $2b = 2$ x^2y^2: $2c + d = 3$ xy^3: $2e = 2$
y^4: $f = 2$

$$X = \begin{pmatrix} 1 & 1 & c \\ 1 & 3-2c & 1 \\ c & 1 & 2 \end{pmatrix} \succeq 0 \iff -1 \le c \le 1$$

$c = -1 \rightsquigarrow p = (x^2 + xy - y^2)^2 + (y^2 + 2xy)^2$

$c = 0 \rightsquigarrow p = (x^2 + xy)^2 + \frac{3}{2}(xy + y^2)^2 + \frac{1}{2}(xy - y^2)^2$

Which nonnegative polynomials are SOS?

Theorem

Hilbert [1888] classified the pairs (n, d) for which every nonnegative polynomial of degree d in n variables is SOS:

- $n = 1$
- $d = 2$
- $n = 2,\ d = 4$

There are nonnegative polynomials that are **not SOS** *in all other cases.*

Example

- **Motzkin polynomial [1967]** $n = 2$, $d = 6$:
  $x^4y^2 + x^2y^4 - 3x^2y^2 + 1$ is nonnegative but **not** a SOS.
- **Choi-Lam polynomial**, $n = 3$, $d = 4$:
  $x^2y^2 + y^2z^2 + x^2z^2 - 4xyz + 1$ is nonnegative, but **not** SOS.

Positivity certificates over a semi-algebraic set K

What about **positivity on K** $= \{x \mid h_1(x) \geq 0, \ldots, h_m(x) \geq 0\}$?

Quadratic module: $\mathcal{M}(h) = \{s_0 + s_1 h_1 + \ldots + s_m h_m \mid s_j \text{ SOS}\}$

Clearly: $p \in \mathcal{M}(h) \implies p \geq 0$ on K.

Theorem (Putinar 1993)

Assume K is **compact** + **Archimedean condition (C)**. *Then,*

$$p > 0 \text{ on } K \implies p \in \mathcal{M}(h).$$

Archimedean condition (C): There exists a constant C such that

$$C^2 - \sum_{i=1}^{n} x_i^2 \in \mathcal{M}(h).$$

Easy to satisfy: add the equation of a ball containing K!

SOS/moment relaxations

Definition (Truncated quadratic module)

$\mathcal{M}(h)_t := \{s_0 + s_1 h_1 + \ldots + s_m h_m \mid s_j \text{ SoS}, \deg(s_j h_j) \leq 2t\}$

Relax

$$p_{\min} = \sup \lambda \text{ s.t. } p - \lambda > 0 \text{ on } K$$

by

$$p_{\text{sos}}^{(t)} := \sup \lambda \text{ s.t. } p - \lambda \in \mathcal{M}(h)_t$$

and define

$$p_{\text{mom}}^{(t)} := \inf_{L \in \mathbb{R}[x]_{2t}^*} L(p) \text{ s.t. } L(1) = 1, \; L \geq 0 \text{ on } \mathcal{M}(h)_t$$

Theorem

- *Weak duality:* $p_{sos}^{(t)} \leq p_{mom}^{(t)} \leq p_{\min}$.
- **Asymptotic convergence:** $\lim_{t\to\infty} p_{sos}^{(t)} = \lim_{t\to\infty} p_{mom}^{(t)} = p_{\min}$.

Properties of the moment relaxations

- Basic idea: **Linearize** the non-linear terms $x^\alpha \rightsquigarrow L(x^\alpha)$.
- $L \in \mathbb{R}[x]_{2t}^* \rightsquigarrow$ **moment (Hankel) matrix of order** t:

$$M_t(L) := (L(x^\alpha x^\beta))_{|\alpha|,|\beta|\leq t}$$

- $h \in \mathbb{R}[x]$, $d_h = \lceil \deg(h)/2 \rceil \rightsquigarrow$ **localizing moment matrix:**

$$M_{t-d_h}(hL) := (L(x^\alpha x^\beta h))_{|\alpha|,|\beta|\leq t-d_h}$$

Lemma (SDP formulation)

- $L(f^2) \geq 0$ *for all* $f \in \mathbb{R}[x]_t \iff M_t(L) \succeq 0.$
- $L(f^2 h) \geq 0$ *whenever* $\deg(f^2 h) \leq 2t \iff M_{t-d_h}(hL) \succeq 0.$

$$p_{mom}^{(t)} = \inf_{L \in \mathbb{R}[x]_{2t}^*} L(p) \quad s.t. \quad L(1) = 1, \; L \geq 0 \text{ on } \mathcal{M}(h)_t$$

can be expressed as a semidefinite program.

Optimality criterion for the moment relaxations

Namely:

$$p_{mom}^{(t)} = \inf_{L \in \mathbb{R}[x]_{2t}^*} L(p) \quad s.t. \quad L(1) = 1, \; M_t(L) \succeq 0,$$

$$M_{t-d_{h_j}}(h_j L) \succeq 0 \; (j = 1, \ldots, m)$$

$$M_t(L) = \begin{array}{|c|c|} \hline M_{t-d_K}(L) & \\ \hline & \\ \hline \end{array} \qquad d_K := \max_j \lceil \deg(h_j)/2 \rceil$$

Theorem (Curto-Fialkow 1996 - Lasserre 2005 - L 2005)

If an optimal solution L satisfies:

$$\mathrm{rank} M_t(L) = \mathrm{rank} M_{t-d_K}(L),$$

then $p_{mom}^{(t)} = p_{\min}$ *and one can compute* **global minimizers.**

Non-commutative polynomial optimization

Extension to NC polynomial optimization

For applications in quantum information, we use polynomials in non-commutative (nc) variables, evaluated at operators:

NC polynomial optimization problem:

$$\textbf{(NCP)} \quad \inf_{H,\Psi,X=(X_1,\ldots,X_n)} \langle \Psi, p(X)\Psi \rangle \quad s.t. \quad h_j(X) \succeq 0 \quad (j \leq m)$$

- H is a Hilbert space (finite or infinite dimension)
- $\Psi \in H$ unit vector
- $X_1, \ldots, X_n \in \mathcal{B}^*(H)$: **bounded Hermitian** operators on H
- $p, h_j \in \mathbb{R}\langle x \rangle = \mathbb{R}\langle x_1, \ldots, x_n \rangle$ are symmetric: $p^* = p$, $h_j^* = h_j$.

$\rightsquigarrow$ **Eigenvalue minimization problem**:

$$\sup \lambda \ s.t. \ p(X) - \lambda I \succeq 0 \ on \ \mathcal{K}_\infty = \bigcup_H \{X \in \mathcal{B}^*(H)^n : h_j(X) \succeq 0 \, \forall j\}.$$

Motivation: non-local games

Two players: Alice and Bob, and the referee.

- The referee chooses a pair of questions $(s, t) \in S \times T$ according to probability $\pi(s, t)$.
- The referee sends question s to Alice and question t to Bob.
- Alice answers $a \in A$, Bob answers $b \in B$, using some **strategy**, chosen before the start of the game.
- Probability of answer (a, b) to question (s, t): $P(a, b|s, t)$.
- Predicate $V : A \times B \times S \times T \to \{0, 1\}$ such that Alice & Bob win the game iff $V(a, b|s, t) = 1$.
- **Value of the game =** Max. probability of winning the game:

$$\max_{P} \sum_{s,t} \pi(s, t) \sum_{a,b} V(a, b|s, t) P(a, b|s, t).$$

Non-local games: Classical strategies

Alice and Bob determine their outputs by employing both *private and shared randomness*:

- Shared random variable: $i \in [n]$ occurs with probability k_i.
- For $i \in [n]$, $s \in S$, Alice has prob. distribution $\{x_a^{s,i} : a \in A\}$.
- For $i \in [n]$, $t \in T$, Bob has a prob. distribution $\{y_b^{t,i} : b \in B\}$.
- **Probability of answer** (a, b): $P(a, b|s, t) = \sum_{i=1}^{n} k_i x_a^{s,i} y_b^{t,i}$.

The classical correlations $(P(a, b|s, t))$ form a **polytope** $\mathcal{P}$:

- The vertices correspond to *deterministic strategies* where Alice selects a function $\alpha : S \to A$, Bob selects function $\beta : T \to B$, and $P(a, b|s, t) = \delta_{a,\alpha(s)}\delta_{b,\beta(t)}$.
- The faces correspond to the **Bell inequalities**.
- **classical value of the game:** $\omega = \max \langle C, P \rangle$ over $P \in \mathcal{P}$.
  $\rightsquigarrow$ **LP**, but hard as the facial structure of $\mathcal{P}$ is **unknown!**

Non-local games: Quantum strategies

- Alice and Bob share an **entangled state** $\Psi \in H_A \otimes H_B$, with H_A, H_B finite dimension.
- $\forall s \in S$ Alice has POVM $\{A_s^a\}_{a \in A}$: $\quad A_s^a \succeq 0, \sum_{a \in A} A_s^a = I_{H_A}$.
- $\forall t \in T$ Bob has POVM $\{B_t^b\}_{b \in B}$: $\quad B_t^b \succeq 0, \sum_{b \in B} B_t^b = I_{H_B}$.
- **Probability of answer** (a, b): $P(a, b|s, t) = \langle \Psi, A_s^a \otimes B_t^b \ \Psi \rangle$.
- **Entangled value of the game:**
$$\omega_q := \sup_{\Psi, A_s^a, B_t^b} \sum_{s,t} \pi(s, t) \sum_{a,b} V(a, b|s, t) \ \langle \Psi, A_s^a \otimes B_t^b \ \Psi \rangle.$$

The quantum correlations $(P(a, b|s, t))$ form a convex *non-polyhedral* set $\mathcal{Q}$. Clearly:

$$\mathcal{P} \subseteq \mathcal{Q}$$

Example: XOR games

For XOR-games:

- $A = B = \{0, 1\}$
- $f : S \times T \longrightarrow \{0, 1\}$
- **Predicate function:** $V(a, b|s, t) = 1$ iff $a \oplus b = f(s, t)$.

Theorem (Tsirelson 1987)

The entangled value ω_q of an XOR-game can be computed efficiently as a semidefinite program.

Special case: CHSH game

- $S = T = \{0, 1\}$ and $f(s, t) = st$.
- Quantum value: $\omega_q = \frac{1}{2} + \frac{1}{2\sqrt{2}} \sim 0.85$, classical val.: $\omega = 0.75$.

So ω_q is easy to compute for XOR games (*the SDP is the first level of the hierarchy to come*), but hard for general games.

Relaxation as NC polynomial optimization problem

$$\omega_q := \sup_{\Psi, A_s^a, B_t^b} \sum_{s,t} \pi(s,t) \sum_{a,b} V(a,b|s,t) \, \langle \Psi, A_s^a \otimes B_t^b \, \Psi \rangle.$$

Remove tensors with **new variables**: $E_s^a = A_s^a \otimes I$, $F_t^b = I \otimes B_t^b$.

- $A_s^a \otimes B_t^b = E_s^a F_t^b$, thus $\langle \Psi, A_s^a \otimes B_t^b \, \Psi \rangle = \langle \Psi, E_s^a F_t^b \, \Psi \rangle$.
- **POVM's**: $E_s^a, F_t^b \succeq 0$, $\sum_{a \in A} E_s^a = I$, $\sum_{b \in B} F_t^b = I \quad \forall s, t$
- **'Bipartite' commutativity:** $[E_s^a, F_t^b] = 0 \quad \forall a, b, s, t$
- **'Commuting' entangled value:** it is expressed by a NC polynomial optimization problem (allow $\dim H = \infty$):

$$\omega_{qc} := \sup_{\Psi \in H, E_s^a, F_t^b} \sum_{s,t} \pi(s,t) \sum_{a,b} V(a,b|s,t) \, \langle \Psi, E_s^a F_t^b \, \Psi \rangle.$$

- $\mathcal{Q}_c$: set of **'commuting' quantum correlations.**

$$\mathcal{Q} \subseteq \mathcal{Q}_c, \quad \mathcal{Q} \neq \mathcal{Q}_c \text{ [Slofstra'16]}, \quad \omega_q \leq \omega_{qc}.$$

How to solve NC polynomial optimization?

Strategy:
Build a hierarchy of SDP relaxations for NC polynomial optimization using **sums of (Hermitian) squares** and the dual **moment theory**.

Acin-Navascues-Pironio [2007–2010] [moment approach]
Doherty-Liang-Toner-Wehner [2008] [SOS approach]

Applications: Get a hierarchy of approximations for:

- 'Commuting' entangled value of non-local games.
- Maximum quantum violation of Bell inequalities

Here: $x_i^* = x_i$ for all variables.

Unconstrained NC polynomial optimization

$p \in \mathbb{R}\langle x \rangle$ is symmetric if $p^* = p$. E.g. $p = x_1 x_2 x_3 + x_3 x_2 x_1$.

Theorem (Helton 2002, McCullough 2002)

A symmetric polynomial p is **matrix-nonnegative**, *i.e., satisfies*

$$p(X) \succeq 0 \quad \text{for all } X = (X_1, \ldots, X_n) \in \cup_d (\mathcal{S}^d)^n$$

if and only if p is a **sum of Hermitian squares (SOS)**:

$$p = \sum_i f_i^* f_i \quad \text{for some } f_i \in \mathbb{R}\langle x \rangle.$$

Corollary (Unconstrained NC polynomial optimization)

$$\begin{aligned} \inf_{\psi, X} \langle \psi, p(X)\psi \rangle &= \sup \; \lambda \;\; \text{s.t. } p(X) - \lambda I \succeq 0 \;\; \forall X \in \cup_d (\mathcal{S}^d)^n \\ &= \sup \; \lambda \;\; \text{s.t. } p - \lambda \;\; \text{is SOS} \end{aligned}$$

can be solved by a **single SDP**.

Positivity certificate in constrained NC optimization

$h_1, \ldots, h_m \in \mathbb{R}\langle x \rangle$ symmetric polynomials

Positivity region:

$$\mathcal{K}_\infty = \bigcup_H \{X = (X_1, \ldots, X_n) \in \mathcal{B}^*(H)^n : h_j(X) \succeq 0 \ \forall j\}$$

$$\mathcal{K} = \bigcup_d \{X = (X_1, \ldots, X_n) \in (\mathcal{S}^d)^n : h_j(X) \succeq 0 \ \forall j\} \subseteq \mathcal{K}_\infty.$$

Quadratic module:

$$\mathcal{M}(h) = \{\textstyle\sum_i f_i^* f_i + \sum_j \sum_i g_{ij}^* h_j g_{ij} \mid f_i, g_{ij} \in \mathbb{R}\langle x \rangle\}$$

Clearly:

$$p \in \mathcal{M}(h) \implies p(X) \succeq 0 \quad \forall X \in \mathcal{K}_\infty.$$

Theorem (Helton-McCullough 2004)

Under Archimedean condition (C): $R - \sum_i x_i^2 \in \mathcal{M}(h)$ for some R,

$$p(X) \succ 0 \quad \forall X \in \mathcal{K}_\infty \implies p \in \mathcal{M}(h).$$

SOS approach to NC optimization

Given symmetric polynomials $p, h_j \in \mathbb{R}\langle x\rangle$

$$\text{(NCP)} \qquad p_{\min} := \inf_{\psi, X} \langle \Psi, p(X)\Psi\rangle \ \ s.t. \ \ h_j(X) \succeq 0 \ (j \le m)$$

$$= \sup \ \lambda \ \ s.t. \ \ p - \lambda \succeq 0 \ \ on \ \ \mathcal{K}_\infty$$

Truncated quadratic module:

$$\mathcal{M}(h)_t = \{\underbrace{\sum_i f_i^* f_i}_{\deg \le 2t} + \sum_j \sum_i \underbrace{g_{ij}^* h_j g_{ij}}_{\deg \le 2t} \mid f_i, g_{ij} \in \mathbb{R}\langle x\rangle\}$$

(SOS relaxation)

$p_{sos}^{(t)} := \sup \ \lambda \ \ s.t. \ \ p - \lambda \in \mathcal{M}(h)_t.$

Theorem (Asymptotic convergence (NPA, DLTW))

Under Archimedean condition (C): $\lim_{t\to\infty} p_{sos}^{(t)} = p_{\min}.$

based on [Helton-McCullough'04].

Finite convergence for the ball and hypercube

- The matrix-ball: $h = 1 - \sum_{i=1}^{n} x_i^2$
- The matrix-hypercube: $h_j = 1 - x_j^2$ for $j = 1, \ldots, n$,

Theorem (Cafuta-Klep-Povh 2012)

$\mathcal{K}$ is the matrix-ball or the matrix-hypercube.
Let p be a symmetric polynomial of degree d.

1. *Characterization of* **nonnegativity** *with* **degree bound***:*

$$p(X) \succeq 0 \text{ on } \mathcal{K} \iff p \in \mathcal{M}(h)_{d+1} \iff p(X) \succeq 0 \text{ on } \mathcal{K}_\infty.$$

2. **Finite convergence***: The parameter*

$$p_{\min} := \inf_{\psi, X} \langle \psi, p(X)\psi \rangle \quad s.t. \; X \in \mathcal{K} \quad (or \; \mathcal{K}_\infty)$$

can be computed via a **single SDP***:*

$$p_{\min} = \sup \; \lambda \quad s.t. \quad p - \lambda \in \mathcal{M}(h)_{d+1}.$$

Moment approach to NC optimization

Given a solution (X, Ψ) to (NCP), define the linear functional $L : \mathbb{R}\langle x\rangle_{2t} \to \mathbb{R}$ by $L(p) = \langle \Psi, p(X)\Psi\rangle$. Then:

- $L(1) = 1$, L is **symmetric**: $L(f) = L(f^*)$ for $f \in \mathbb{R}\langle x\rangle_{2t}$,
- L is **nonnegative**: $L(f) \geq 0$ for all $f \in \mathcal{M}(h)_t$.

(Moment relaxation)

- $p^{(t)}_{mom} = \min_{L\in\mathbb{R}\langle x\rangle^*_{2t},\ sym.} L(p)$ *s.t.* $L(1) = 1$, $L \geq 0$ *on* $\mathcal{M}(h)_t$.
- $p^{(t)}_{sos} \leq p^{(t)}_{mom} \leq p_{\min}$.

Theorem (Asymptotic convergence & optimality criterion, NPA)

1. *Under (C),* $\lim_{t\to\infty} p^{(t)}_{mom} = p_{\min}$ *& one can construct an optimum solution* (H, ψ, X) *to* $p_{\min}$.
2. *If an optimal sol.* L *is* **flat**: $\mathrm{rank} M_t(L) = \mathrm{rank} M_{t-d_K}(L) =: r$, *then the relaxation is exact:* $p^{mom}_t = p_{\min}$ *and one can construct an optimal sol.* (H, X, Ψ) *with* **finite** $\dim H = r$.

Sketch of proof

Theorem (Optimality criterion, Acin-Navascues-Pironio 2010)

If an optimal solution L is **flat***:* $\mathrm{rank} M_t(L) = \mathrm{rank} M_{t-d_K}(L) =: r$*, then the relaxation is exact:* $p_t^{mom} = p_{\min}$ *and one can construct a solution* (H, X, Ψ) *of (NCP) with* $\dim H = r$*.*

- Say, $M_t(L)$ is the Gram matrix of vectors $\overline{w}$ ($w \in \mathcal{W}_t$).
- That is, $L(v^* w) = \langle \overline{v^*}, \overline{w} \rangle$ for $v, w \in \mathcal{W}_t$.
- Let $H = \mathrm{Span}\{\overline{w} \mid w \in \mathcal{W}_t\} = \mathrm{Span}\{\overline{w} \mid w \in \mathcal{W}_{t-d_K}\}$.
- Define the operators $X_i : H \to H$ by

$$X_i \overline{v} = \overline{x_i v} \quad \text{for } v \in \mathcal{W}_{t-d_K}$$

- **To show:** $X_1, \ldots, X_n$ and $\Psi := \overline{1}$ give a solution of (NCP).
- In the **commutative** case, X_i commute pairwise, so can be simultaneously diagonalized, and the eigenvalues give the scalar solutions.

Application to the entangled value ω_q

Corollary (NPA)

1. *The* **'commuting' entangled value** ω_{qc} *can be obtained as the limit of a hierarchy of SDP's, using NC polynomial optimization.*
2. *If the optimal solution L of some moment relaxation is* **flat** *then one finds in fact the* **entangled value** ω_q.

Used by Pal-Vertesi, Navascues, ...

Another extension: tracial nc polynomial optimization

(Tracial optimization (Burgdorf, Cafuta, Klep, Povh))

$p^{tr}_{min} := \inf_{H,X} \ \mathrm{tr}(p(X_1, \ldots, X_n)) \ \ s.t. \ \ h_j(X) \succeq 0 \ \forall j.$

Note: The trace vanishes on **commutators:** $[f, g] = fg - gf$

Strengthen the moment/SOS relaxations:

1. Add constraint: L is **tracial**: $L(fg) = L(gf)$
2. Use commutators to decompose: $p - \lambda \in \mathcal{M}(h) + \{\sum_i [f_i, g_i]\}$

But the parameter $p^{tr}_{\min}$ is more difficult to approach...

Theorem (Klep-Povh 2016)

Under the Archimedean condition, the tracial SOS/moment relaxations converge to the **variation** *of p^{tr}_{min}, where we allow* **solutions in any finite von Neumann algebra***.*

Link to Connes' embedding conjecture

Consider the assertions:

(1) $\forall \epsilon > 0 \quad p + \epsilon \in \mathcal{M}(1 - x_1^2, \ldots, 1 - x_n^2) + \{\sum_i [f_i, g_i]\}$

(2) $\tau(p(x)) \geq 0$ on all *contractions in any finite von Neumann algebra with trace* τ.

(3) $Tr(p(X)) \geq 0$ on the matrix-hypercube (all contraction matrices)

Clearly: (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3).

Theorem (Klep-Schweighofer 2008)

1. *(1) and (2) are equivalent.*
2. *Connes' embedding conjecture holds*
   $\Longleftrightarrow$ *(1) and (3) are equivalent for any* p.

Final look at quantum correlations

Theorem (Mancinska-Roberson'14, Sikora-Varvitsiotis'15)

$P = (P(a,b|s,t)) \in \mathcal{Q}$ *if and only if*

there exist $E_s^a, F_t^b \succeq 0$, $K \succeq 0$ *such that*

- $Tr(K^2) = 1$.
- $\sum_a E_s^a = \sum_b F_t^b = K$ *for all* s, t.
- $P(a,b|s,t) = Tr(E_s^a F_t^b)$ *for all* a, b, s, t.